

Cosmology in Weyl Transverse Gravity

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Abstract

We study the Friedmann-Lemaitre-Robertson-Walker (FLRW) cosmology in the Weyl-transverse (WTDiff) gravity in a general space-time dimension. The WTDiff gravity is invariant under both the local Weyl (conformal) transformation and the volume preserving diffeomorphisms (transverse diffeomorphisms) and is believed to be equivalent to general relativity at least at the classical level (perhaps, even in the quantum regime). It is explicitly shown by solving the equations of motion that the FLRW metric is a classical solution in the WTDiff gravity only when the spatial metric is flat, that is, the Euclidean space, and the lapse function is a nontrivial function of the scale factor.

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1 Introduction

In a series of recent papers [1]-[3], we have investigated some classical implications of a gravitational model called the Weyl-transverse (WTDiff) gravity [4]-[9], which is invariant under both the local Weyl (conformal) transformation and a restricted subgroup of general coordinate transformation or diffeomorphisms (Diff), that is, the volume preserving diffeomorphisms or transverse diffeomorphisms (TDiff). We have already demonstrated that general relativity and the WTDiff gravity are obtained via gauge fixing procedure for a different local symmetry from the conformally invariant scalar-tensor gravity ², which is a more underlying model in the sense that the conformally invariant scalar-tensor gravity is invariant under both the Weyl transformation and the full group of Diff. Indeed, general relativity is obtained by gauge-fixing the Weyl symmetry while the WTDiff gravity is reached by doing the longitudinal diffeomorphism from the conformally invariant scalar-tensor gravity.

This relation between general relativity and the WTDiff gravity gives us an interesting observation that the two theories are equivalent to each other although they have different local symmetries so that they belong to different universality classes. In fact, we have shown that in the WTDiff gravity the Schwarzschild metric and the Reissner-Nordstrom metric in the Cartesian coordinate system are classical solutions to the equations of motion as in general relativity, but they are not so in the other coordinate systems like the spherical coordinate one. This peculiar dependence of classical solutions on the coordinate systems in the WTDiff gravity stems from the fact that the TDiff are defined as a subgroup of the full Diff in such a way that the determinant of the transformation matrix is the unity

$$J \equiv \det J_{\mu}^{\alpha} \equiv \det \frac{\partial x^{\alpha}}{\partial x^{\mu}} = 1. \quad (1)$$

When we transform a metric from the Cartesian coordinate system to the spherical one, we encounter the non-trivial Jacobian factor in four dimensions

$$J = r^2 \sin \theta, \quad (2)$$

by which the Schwarzschild and Reissner-Nordstrom metrics are not classical solutions in the spherical coordinate system in the WTDiff gravity.

The equivalence between general relativity and the WTDiff gravity might make it possible to tackle some difficult problems existing in general relativity within the framework of the WTDiff gravity. For instance, the Weyl symmetry in the WTDiff gravity can be viewed as a *fake* Weyl symmetry, ³ which might play an important role in the cosmological constant

²The conformally invariant gravity theory has a wide application in phenomenology and cosmology [10]-[13].

³The Weyl symmetry in the conformally-invariant scalar-tensor gravity as well as the WTDiff gravity is sometimes called a *fake* Weyl symmetry [14] since this Weyl symmetry appears as a local symmetry in an action whenever one replaces the metric tensor $g_{\mu\nu}$ with a Weyl-invariant metric tensor $\hat{g}_{\mu\nu}$ as seen later in

problem. Namely, at the classical level, the *fake* Weyl symmetry forbids operators of dimension zero such as the cosmological constant in the action. Then, we expect that the *fake* Weyl symmetry does not give rise to a Weyl anomaly at the quantum level owing to its "fakeness" [1]. In other words, the *fake* Weyl symmetry survives at the quantum level, thereby suppressing the radiative corrections to the cosmological constant. If this conjecture were true, the cosmological constant problem would become a mere problem of how to determine the initial value of the cosmological constant.

In the present article, in order to understand more classical implications of the WTDiff gravity, we wish to apply the WTDiff gravity to cosmology and find what universe is realized in the WTDiff gravity by solving the equations of motion on the basis of Friedmann-Lemaitre-Robertson-Walker (FLRW) metric. To do that, one needs to introduce a real scalar field as the matter field which must be invariant under the Weyl transformation and TDiff at the same time.

This paper is organised as follows: In Section 2, we couple a real scalar field as the matter field to the WTDiff gravity in a consistent way, and examine the energy-momentum tensor and its conservation law. In Section 3, we show that the FLRW metric with the spatial flat metric and a nontrivial lapse function is in fact a classical solution. The final section is devoted to discussions.

2 Coupling scalar field to the Weyl-transverse (WTDiff) gravity

We will start with an action of the conformally invariant scalar-tensor gravity in a general n space-time dimension ⁴

$$S = \int d^n x \sqrt{-g} \left[\frac{n-2}{8(n-1)} \varphi^2 R + \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \right], \quad (3)$$

Eq. (9). For instance, the conformally-invariant scalar-tensor gravity is obtained by replacing $g_{\mu\nu}$ in the Einstein-Hilbert action by $\hat{g}_{\mu\nu}$ (In this sense, φ is called a spurion field.) The "fakeness" is mathematically reflected in the fact that the Noether current for this Weyl symmetry is identically vanishing [14, 1].

⁴We follow notation and conventions by Misner et al.'s textbook [15], for instance, the flat Minkowski metric $\eta_{\mu\nu} = \text{diag}(-, +, +, +)$, the Riemann curvature tensor $R^\mu{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^\mu_{\nu\beta} - \partial_\beta \Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\sigma\alpha} \Gamma^\sigma_{\nu\beta} - \Gamma^\mu_{\sigma\beta} \Gamma^\sigma_{\nu\alpha}$, and the Ricci tensor $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$. The reduced Planck mass is defined as $M_p = \sqrt{\frac{c\hbar}{8\pi G}} = 2.4 \times 10^{18} \text{GeV}$. Throughout this article, we adopt the reduced Planck units where we set $c = \hbar = M_p = 1$. In this units, all quantities become dimensionless. Finally, note that in the reduced Planck units, the Einstein-Hilbert Lagrangian density takes the form $\mathcal{L}_{EH} = \frac{1}{2} \sqrt{-g} R$.

which is invariant under both the Weyl transformation and diffeomorphisms (Diff). The Weyl transformation is defined for the metric tensor $g_{\mu\nu}$ and the ghost-like scalar field φ as

$$g_{\mu\nu} \rightarrow g'_{\mu\nu} = \Omega^2(x)g_{\mu\nu}, \quad \varphi \rightarrow \varphi' = \Omega^{-\frac{n-2}{2}}(x)\varphi, \quad (4)$$

where $\Omega(x)$ is an arbitrary scalar function.

From this fundamental action (3), we can derive the Einstein-Hilbert action of general relativity by taking the gauge condition

$$\varphi = 2\sqrt{\frac{n-1}{n-2}}, \quad (5)$$

for the Weyl symmetry.

On the other hand, the gauge condition

$$\varphi = 2\sqrt{\frac{n-1}{n-2}}|g|^{-\frac{n-2}{4n}}, \quad (6)$$

for the longitudinal diffeomorphism leads to an action of the WTDiff gravity [6, 7, 8, 9, 2, 3]

$$S_g = \frac{1}{2} \int d^n x |g|^{\frac{1}{n}} \left[R + \frac{(n-1)(n-2)}{4n^2} \frac{1}{|g|^2} g^{\mu\nu} \partial_\mu |g| \partial_\nu |g| \right], \quad (7)$$

where we have defined $g = \det g_{\mu\nu} < 0$. Thus, the WTDiff gravity is at least classically equivalent to general relativity since the both actions can be derived via the different choices of gauge condition from the same action (3). We conjecture that the equivalence would be valid even at the quantum level since the *fake* Weyl symmetry and the longitudinal diffeomorphism do not seem to possess the corresponding anomalies.

Now we wish to construct the matter action of a real scalar field ϕ coupled to gravity in the Weyl and TDiff-invariant manner [1]. To do so, let us consider first an action of the real scalar field ϕ with the potential term $V(\phi)$

$$S_m^{Diff} = \int d^n x |g|^{\frac{1}{2}} \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]. \quad (8)$$

Note that S_m^{Diff} is manifestly invariant under the full Diff.

Next, to make S_m^{Diff} be invariant under the Weyl transformation as well, it is necessary to construct the Weyl-invariant objects

$$\hat{g}_{\mu\nu} = \left(\frac{1}{2} \sqrt{\frac{n-2}{n-1}} \varphi \right)^{\frac{4}{n-2}} g_{\mu\nu}, \quad \hat{\phi} = \frac{\phi}{\varphi}, \quad (9)$$

where we have assumed that the real scalar field ϕ has the same transformation property under the Weyl transformation (4) as the ghost-like scalar field φ . In order to obtain the

Weyl and Diff-invariant matter action, it is enough to replace $g_{\mu\nu}$ and ϕ in the action (8) by the corresponding Weyl-invariant objects $\hat{g}_{\mu\nu}$ and $\hat{\phi}$. The resulting action takes the form

$$\begin{aligned}
S_m^{WDiff} &= \int d^n x |\hat{g}|^{\frac{1}{2}} \left[-\frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \hat{\phi} \partial_\nu \hat{\phi} - V(\hat{\phi}) \right] \\
&= \int d^n x |g|^{\frac{1}{2}} \left[-\frac{1}{2} \left(\frac{1}{2} \sqrt{\frac{n-2}{n-1}} \varphi \right)^2 g^{\mu\nu} \partial_\mu \left(\frac{\phi}{\varphi} \right) \partial_\nu \left(\frac{\phi}{\varphi} \right) \right. \\
&\quad \left. - \left(\frac{1}{2} \sqrt{\frac{n-2}{n-1}} \varphi \right)^{\frac{2n}{n-2}} V \left(\frac{\phi}{\varphi} \right) \right].
\end{aligned} \tag{10}$$

Finally, reducing further the Weyl and Diff-invariant matter action (10) to the Weyl and TDiff-invariant matter action requires us to take the gauge condition (6) for the longitudinal diffeomorphism. Consequently, we have the WTDiff-invariant matter action given by

$$\begin{aligned}
S_m^{WTDiff} &= \int d^n x |g|^{\frac{1}{2}} \left\{ -\frac{1}{8} \frac{n-2}{n-1} g^{\mu\nu} \left[\left(\frac{n-2}{4n} \right)^2 \frac{\phi^2}{|g|^2} \partial_\mu |g| \partial_\nu |g| + \frac{n-2}{2n} \frac{\phi}{|g|} \partial_\mu |g| \partial_\nu \phi + \partial_\mu \phi \partial_\nu \phi \right] \right. \\
&\quad \left. - |g|^{-\frac{1}{2}} V \left(\frac{1}{2} \sqrt{\frac{n-2}{n-1}} |g|^{\frac{n-2}{4n}} \phi \right) \right\}.
\end{aligned} \tag{11}$$

From this action, the equation of motion for ϕ is derived to be

$$\begin{aligned}
&\frac{1}{8} \frac{n-2}{n-1} |g|^{\frac{1}{2}} \left[\frac{(n-2)(5n-2)}{8n^2} \frac{\phi}{|g|^2} (\partial_\rho |g|)^2 - \frac{n-2}{2n} \frac{\phi}{|g|} \nabla_\rho \nabla^\rho |g| - 2 \nabla_\rho \nabla^\rho \phi \right] \\
&+ \frac{1}{2} \sqrt{\frac{n-2}{n-1}} |g|^{\frac{n-2}{4n}} V' \left(\frac{1}{2} \sqrt{\frac{n-2}{n-1}} |g|^{\frac{n-2}{4n}} \phi \right) \\
&= 0,
\end{aligned} \tag{12}$$

where we have defined $\nabla_\mu \nabla_\nu |g| = \partial_\mu \partial_\nu |g| - \Gamma_{\mu\nu}^\rho \partial_\rho |g|$ and the prime on the potential V denotes the differentiation with respect to its argument. Furthermore, taking the variation of the action (11) with respect to the metric tensor produces an expression, which is proportional to the energy-momentum tensor of the scalar matter field

$$\begin{aligned}
\frac{\delta S_m^{WTDiff}}{\delta g^{\mu\nu}} &= -\frac{1}{8} \frac{n-2}{n-1} |g|^{\frac{1}{2}} \left\{ \left(\frac{n-2}{4n} \right)^2 \frac{\phi^2}{|g|^2} \partial_\mu |g| \partial_\nu |g| + \frac{n-2}{4n} \frac{\phi}{|g|} (\partial_\mu |g| \partial_\nu \phi + \partial_\nu |g| \partial_\mu \phi) \right. \\
&\quad + \partial_\mu \phi \partial_\nu \phi + g_{\mu\nu} \left[-\frac{5}{2} \left(\frac{n-2}{4n} \right)^2 \frac{\phi^2}{|g|^2} (\partial_\rho |g|)^2 - \frac{n-2}{2n^2} \frac{\phi}{|g|} \partial_\rho \phi \partial^\rho |g| - \frac{1}{n} (\partial_\rho \phi)^2 \right. \\
&\quad \left. \left. + 2 \left(\frac{n-2}{4n^2} \right)^2 \frac{\phi^2}{|g|} \nabla_\rho \nabla^\rho |g| + \frac{n-2}{2n} \phi \nabla_\rho \nabla^\rho \phi \right] \right\} \\
&+ \frac{n-2}{8n} \sqrt{\frac{n-2}{n-1}} |g|^{\frac{n-2}{4n}} g_{\mu\nu} \phi V' \left(\frac{1}{2} \sqrt{\frac{n-2}{n-1}} |g|^{\frac{n-2}{4n}} \phi \right).
\end{aligned} \tag{13}$$

Then, it is convenient to rewrite this expression by eliminating the potential term through the equation of motion for ϕ in Eq. (12) as

$$\begin{aligned}\frac{\delta S_m^{WTDiff}}{\delta g^{\mu\nu}} &= -\frac{1}{8} \frac{n-2}{n-1} |g|^{\frac{1}{2}} \left\{ \left(\frac{n-2}{4n} \right)^2 \frac{\phi^2}{|g|^2} \partial_\mu |g| \partial_\nu |g| + \frac{n-2}{4n} \frac{\phi}{|g|} (\partial_\mu |g| \partial_\nu \phi + \partial_\nu |g| \partial_\mu \phi) \right. \\ &\quad \left. + \partial_\mu \phi \partial_\nu \phi + g_{\mu\nu} \left[-\frac{1}{n} \left(\frac{n-2}{4n} \right)^2 \frac{\phi^2}{|g|^2} (\partial_\rho |g|)^2 - \frac{n-2}{2n^2} \frac{\phi}{|g|} \partial_\rho \phi \partial^\rho |g| - \frac{1}{n} (\partial_\rho \phi)^2 \right] \right\} \\ &\equiv -\frac{1}{2} |g|^{\frac{1}{2}} \left(T_{(m)\mu\nu} - \frac{1}{n} g_{\mu\nu} T_{(m)} \right),\end{aligned}\tag{14}$$

where $T_{(m)\mu\nu}$ is defined as

$$\begin{aligned}T_{(m)\mu\nu} &= \frac{1}{4} \frac{n-2}{n-1} \left[\left(\frac{n-2}{4n} \right)^2 \frac{\phi^2}{|g|^2} \partial_\mu |g| \partial_\nu |g| + \frac{n-2}{4n} \frac{\phi}{|g|} (\partial_\mu |g| \partial_\nu \phi + \partial_\nu |g| \partial_\mu \phi) \right. \\ &\quad \left. + \partial_\mu \phi \partial_\nu \phi \right].\end{aligned}\tag{15}$$

Without the matter action of the scalar field, from the action (7) of the WTDiff gravity, the Einstein equations, which are obtained by taking the variation with respect to the metric tensor, take the form

$$R_{\mu\nu} - \frac{1}{n} g_{\mu\nu} R = T_{(g)\mu\nu} - \frac{1}{n} g_{\mu\nu} T_{(g)},\tag{16}$$

where $T_{(g)\mu\nu}$ is defined by

$$T_{(g)\mu\nu} = \frac{(n-2)(2n-1)}{4n^2} \frac{1}{|g|^2} \partial_\mu |g| \partial_\nu |g| - \frac{n-2}{2n} \frac{1}{|g|} \nabla_\mu \nabla_\nu |g|.\tag{17}$$

Since we regard the sum of the action (7) of the WTDiff gravity plus that (10) of the scalar matter field as a total action, the Einstein equations at hand are given by

$$R_{\mu\nu} - \frac{1}{n} g_{\mu\nu} R = T_{\mu\nu} - \frac{1}{n} g_{\mu\nu} T,\tag{18}$$

where the total energy-momentum tensor $T_{\mu\nu}$ is now defined by

$$\begin{aligned}T_{\mu\nu} &= T_{(g)\mu\nu} + T_{(m)\mu\nu} \\ &= \frac{(n-2)(2n-1)}{4n^2} \frac{1}{|g|^2} \partial_\mu |g| \partial_\nu |g| - \frac{n-2}{2n} \frac{1}{|g|} \nabla_\mu \nabla_\nu |g| \\ &\quad + \frac{1}{4} \frac{n-2}{n-1} \left[\left(\frac{n-2}{4n} \right)^2 \frac{\phi^2}{|g|^2} \partial_\mu |g| \partial_\nu |g| + \frac{n-2}{4n} \frac{\phi}{|g|} (\partial_\mu |g| \partial_\nu \phi + \partial_\nu |g| \partial_\mu \phi) \right. \\ &\quad \left. + \partial_\mu \phi \partial_\nu \phi \right].\end{aligned}\tag{19}$$

Let us note that the Einstein equations of the WTDiff gravity with the WTDiff-invariant matter field have the *traceless* form, which is also a common feature to unimodular gravity [16]-[31].

3 Cosmology

Now we are ready to turn our attention to cosmology in the WTDiff gravity with the scalar matter whose equations of motion are of form (18). Before attempting to solve the Einstein equations (18), let us notice that as given in Eq. (19), the energy-momentum tensor has a rather complicated structure owing to the presence of the determinant of the metric tensor, which makes it difficult to solve the Einstein equations analytically. Thus, we will select the gauge condition

$$g = -1, \quad (20)$$

for the Weyl symmetry. This choice of the gauge condition provides us with an enormous simplification since the energy-momentum tensor (19) is reduced to the tractable form

$$T_{\mu\nu} = \frac{1}{4} \frac{n-2}{n-1} \partial_\mu \phi \partial_\nu \phi. \quad (21)$$

It is usually assumed that our universe is described in terms of an expanding, homogeneous and isotropic Friedmann-Lemaitre-Robertson-Walker (FLRW) universe given by the line element

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -dt^2 + a^2(t) \gamma_{ij}(x) dx^i dx^j, \end{aligned} \quad (22)$$

where $a(t)$ is a scale factor and $\gamma_{ij}(x)$ is the spatial metric of the unit $(n-1)$ -sphere, unit $(n-1)$ -hyperboloid or $(n-1)$ -plane, and i, j run over spatial coordinates ($i = 1, 2, \dots, n-1$). However, this metric ansatz does not satisfy the gauge condition (20) so the line element (22) should be somewhat modified. A suitable modification, which respects the gauge condition (20), is to work with the following line element;

$$ds^2 = -N^2(t) dt^2 + a^2(t) (dx^i)^2, \quad (23)$$

where $N(t)$ is a lapse function and the spatial geometry is chosen to be the $(n-1)$ -plane, i.e., the $(n-1)$ -dimensional Euclidean space. Note that the existence of the lapse function $N(t)$ means that a time coordinate t does not coincide with the proper time of particles at rest. With this line element, the gauge condition (20) provides a relation between the lapse function $N(t)$ and the scale factor $a(t)$

$$N(t) = a^{-(n-1)}(t). \quad (24)$$

Given the line element (23) and Eq. (24), it turns out that the non-vanishing components of the *traceless* Einstein tensor, which is defined as

$$G_{\mu\nu}^T = R_{\mu\nu} - \frac{1}{n} g_{\mu\nu} R, \quad (25)$$

are given by

$$\begin{aligned} G_{tt}^T &= -\frac{(n-1)(n-2)}{n} [\dot{H} + (n-1)H^2], \\ G_{ij}^T &= -\frac{n-2}{n} a^{2n} [\dot{H} + (n-1)H^2] \delta_{ij}, \end{aligned} \quad (26)$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter and we have defined $\dot{a} = \frac{da(t)}{dt}$. In a similar way, the non-vanishing components of the *traceless* energy-momentum tensor, which is defined as

$$T_{\mu\nu}^T = T_{\mu\nu} - \frac{1}{n} g_{\mu\nu} T, \quad (27)$$

read

$$\begin{aligned} T_{tt}^T &= \frac{n-2}{4n} (\dot{\phi})^2, \\ T_{ij}^T &= \frac{1}{n-1} \frac{n-2}{4n} a^{2n} (\dot{\phi})^2 \delta_{ij}, \end{aligned} \quad (28)$$

where we have specified the scalar field ϕ to be spatially homogeneous, that is, $\phi = \phi(t)$. As a result, the *traceless* Einstein equations (18) are cast to be a single equation

$$\dot{H} + (n-1)H^2 = -\frac{1}{4(n-1)} (\dot{\phi})^2. \quad (29)$$

Moreover, using the line element (23) and Eq. (24), the equation of motion for the scalar field ϕ , Eq. (12), is simplified to be

$$\ddot{\phi} + 2(n-1)H\dot{\phi} + 2\sqrt{\frac{n-1}{n-2}} a^{-2(n-1)} V' \left(\frac{1}{2} \sqrt{\frac{n-2}{n-1}} \phi \right) = 0. \quad (30)$$

It is of interest to see that the *traceless* Einstein equations (18) have yielded only the single equation (29), which is similar to the Raychaudhuri equation or the first Friedmann equation [32, 33] that comes from all *ij*-components of the Einstein equations in general relativity though there is a slight difference in Eq. (29) which will be mentioned shortly. However, in the present formalism, the (second) Friedmann equation stemming from 00-component of the Einstein equations is missing. In order to solve Eq. (29), we need the conservation law of the energy-momentum tensor. In this respect, recall that in general relativity the first Friedmann equation can be viewed as a consequence of the (second) Friedmann equation and covariant conservation of energy, so that the combination of the (second) Friedmann equation and the conservation law, supplemented by the equation of state $p = p(\rho)$ (which will appear later), forms a complete system of equations that determines the two unknown functions, the scale factor $a(t)$ and energy density ρ . In our formalism, instead of the (second) Friedmann equation, we have to use the first Friedmann equation like Eq. (29).

At this stage, it is worth stressing that the energy-momentum tensor (19) is not covariantly conserved since it is derived from the action which is not invariant under the general coordinate transformation (Diff) but only invariant under the Weyl transformation and TD-iff. Actually, the following (well-known) proof clarifies the reason why the energy-momentum tensor constructed out of a Diff-invariant action is only covariantly conserved: Suppose that a generic action S is invariant under Diff

$$S = \int d^n x \sqrt{-g} \mathcal{L}. \quad (31)$$

Under Diff, the metric tensor transforms as

$$\delta g^{\mu\nu} = \nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu, \quad (32)$$

where ξ^μ is a local parameter of Diff. Under Diff, the action S is transformed into

$$\delta S = - \int d^n x \sqrt{-g} T_{\mu\nu} \nabla^\mu \xi^\nu, \quad (33)$$

where the energy-momentum tensor $T_{\mu\nu}$ is defined as

$$\begin{aligned} T_{\mu\nu} &= - \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L})}{\delta g^{\mu\nu}} \\ &= -2 \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} + g_{\mu\nu} \mathcal{L}. \end{aligned} \quad (34)$$

By integrating by parts, Eq. (33) can be recast to the form

$$\delta S = \int d^n x \sqrt{-g} \nabla_\mu T^{\mu\nu} \xi_\nu, \quad (35)$$

from which we can arrive at the covariant conservation law of the energy-momentum tensor

$$\nabla_\mu T^{\mu\nu} = 0. \quad (36)$$

Let us note that the general coordinate invariance of the action plays a critical role in this proof.

Accordingly, in order to derive the energy-momentum tensor satisfying the covariant conservation law, we must make use of not S_m^{WTDiff} in (11) but S_m^{WDiff} in (10).⁵ After a straightforward calculation, using the gauge condition (20), the energy-momentum tensor reads

$$T_{\mu\nu}^{(cov)} = \frac{1}{4} \frac{n-2}{n-1} \partial_\mu \phi \partial_\nu \phi + g_{\mu\nu} \left[-\frac{1}{8} \frac{n-2}{n-1} (\partial_\rho \phi)^2 - V \left(\frac{1}{2} \sqrt{\frac{n-2}{n-1}} \phi \right) \right]. \quad (37)$$

⁵The contribution from S in (3) vanishes in the gauge condition (20).

In contrast to the previous result (21), in this case the terms proportional to $g_{\mu\nu}$ have emerged, by which the energy-momentum tensor (37) turns out to be covariantly conserved by using the equation of motion for ϕ in Eq. (12) together with the gauge condition (20). The non-vanishing components of $T^{(cov)\mu}{}_{\nu}$ are easily evaluated to be

$$\begin{aligned} T^{(cov)t}{}_t &= -\frac{1}{8}\frac{n-2}{n-1}a^{2(n-1)}(\dot{\phi})^2 - V\left(\frac{1}{2}\sqrt{\frac{n-2}{n-1}}\phi\right) \equiv -\rho(t), \\ T^{(cov)i}{}_j &= \left[\frac{1}{8}\frac{n-2}{n-1}a^{2(n-1)}(\dot{\phi})^2 - V\left(\frac{1}{2}\sqrt{\frac{n-2}{n-1}}\phi\right)\right]\delta^i{}_j \equiv p(t)\delta^i{}_j, \end{aligned} \quad (38)$$

where we have introduced energy density $\rho(t)$ and pressure $p(t)$ in a conventional way. Then, the covariant conservation law (36) leads to an equation

$$\dot{\rho} + (n-1)H(\rho + p) = 0. \quad (39)$$

To close the system of equations, which determine the dynamics of homogeneous and isotropic universe, we have to specify the equation of state of matter as usual

$$p = w\rho, \quad (40)$$

where w is a certain constant. Of course, the equation of state is not a consequence of equations of our formalism, but should be determined by matter content in our universe. With the help of Eq. (40), Eq. (39) is exactly solved to be

$$\rho(t) = \rho_0 a^{-(n-1)(w+1)}(t), \quad (41)$$

where ρ_0 is an integration constant. Eqs. (39)-(41) are the same expressions as in general relativity. Now, using Eqs. (38), (40) and (41), our Friedmann equation (29) is rewritten as

$$\dot{H} + (n-1)H^2 = -\frac{w+1}{n-2}\rho_0 a^{-(n-1)(w+3)}. \quad (42)$$

Since it is difficult to find a general solution to this equation (42), we will refer to only special solutions which are physically interesting. Looking at the RHS in Eq. (42), one soon notices that at $w = -1$ and $w = -3$, specific situations occur. Actually, at $w = -1$, Eq. (42) can be exactly integrated to be

$$a(t) = a_0 t^{\frac{1}{n-1}}, \quad (43)$$

where a_0 is an integration constant and this solution describes the decelerating universe in four dimensions owing to $\ddot{a} < 0$.

At the case $w = -3$, Eq. (42) is reduced to the form

$$\dot{H} + (n-1)H^2 = \frac{2}{n-2}\rho_0. \quad (44)$$

This equation includes a special solution describing an exponentially expanding universe

$$a(t) = a_0 e^{H_0 t}, \quad (45)$$

where H_0 is a constant defined as

$$H_0 = \sqrt{\frac{2\rho_0}{(n-1)(n-2)}}. \quad (46)$$

Finally, one can find a special solution such that the scale factor $a(t)$ is the form of polynomial in t

$$a(t) = a_0 t^\alpha, \quad (47)$$

where α is a constant to be determined by the Friedmann equation (42). It is easy to verify that the constant α is given by

$$\alpha = \frac{2}{(n-1)(w+3)}, \quad (48)$$

so that in this case the scale factor takes the form

$$a(t) = a_0 t^{\frac{2}{(n-1)(w+3)}}, \quad (49)$$

which includes the solution (43) when $w = -1$. Then, the accelerating universe $\ddot{a}(t) > 0$ requires

$$w < \frac{-3n+5}{n-1}, \quad (50)$$

while the decelerating universe does

$$w > \frac{-3n+5}{n-1}. \quad (51)$$

One might wonder how the obtained solutions are related to solutions in general relativity. In particular, in general relativity we are familiar with the fact that the case $w = -1$ corresponds to the cosmological constant and the solution is then an exponentially expanding universe whereas in our case the corresponding solution belongs to the case $w = -3$, which appears to be strange. But this is just an illusion since we do not use the conventional form (22) of the line element but the line element (23) involving the nontrivial lapse function $N(t)$.

In order to show that our result coincides with that in general relativity, let us focus our attention to the Friedmann equation (29). By means of Eq. (38), this equation is rewritten as

$$\dot{H} + (n-1)H^2 = -\frac{1}{n-2}N^2(\rho + p), \quad (52)$$

where we recovered the lapse function $N(t)$ by using Eq. (24).

On the other hand, with the conventional notation of the energy-momentum tensor

$$T^\mu{}_\nu = \text{diag}(-\rho, p, \dots, p), \quad (53)$$

and the line element (23), the Einstein equations in general relativity

$$G^\mu{}_\nu \equiv R^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu R = T^\mu{}_\nu, \quad (54)$$

become a set of the Friedmann equations

$$H^2 = \frac{2}{(n-1)(n-2)}N^2\rho, \quad (55)$$

$$\dot{H} + \frac{n-1}{2}H^2 - \frac{\dot{N}}{N}H = -\frac{1}{n-2}N^2p. \quad (56)$$

By using Eq. (24), Eq. (56) is written as

$$\dot{H} + \frac{3(n-1)}{2}H^2 = -\frac{1}{n-2}N^2p. \quad (57)$$

Eq. (55) allows us to rewrite this equation to the form

$$\dot{H} + (n-1)H^2 = -\frac{1}{n-2}N^2(\rho + p), \quad (58)$$

which coincides with our Friedmann equation (52). This demonstration clearly indicates that our cosmological solution is just equivalent to that of general relativity specified in such a way that the line element is (23) and the lapse function is given by Eq. (24).

4 Discussions

In this article, we have studied the Friedmann-Lemaitre-Robertson-Walker (FLRW) cosmology in the framework of the Weyl-transverse (WTDiff) gravity in a general space-time dimension. One of interesting aspects of our result is that spatial geometry is completely selected to be a flat Euclidean space among three possibilities, those are, the unit sphere, unit hyperboloid and Euclidean space. In regards to this, let us recall that for both closed and open universes, the spatial curvature can be often be neglected, so one can use the spatially flat metric and this is certainly possible for processes happened at spatial scales much smaller than the curvature radius $a(t)$. Our result insists that the spatial metric must be flat at least in the classical level where the present analysis could be applied.

Furthermore, our result requires that the lapse function, which is usually taken to be 1 by hand, should be a nontrivial function of the scale factor. Note that there is *a priori* no

need for fixing the lapse function to be a certain value. To put differently, there is no need for choosing a time coordinate such that it agrees with proper time of particles at rest since world lines of particles at rest are geodesic even in the line element with the nontrivial lapse function of time coordinate t .

As future problems, we would like to list up two different problems. One problem is to look for a broad class of classical solutions which do not satisfy the gauge condition (20). As seen in (19), the total energy-momentum tensor involves the complicated contribution from the metric determinant and this contribution behaves as if it were the source of a new matter field in the traceless Einstein equations. This fact makes it quite difficult to find classical solutions except the case $g = -1$, or more generally, $g = \text{constant}$.

Another interesting and important problem is to investigate quantum aspect of the present formalism. The most attractive point in the present formalism is the existence of the fake Weyl symmetry, by which the cosmological constant cannot appear in the classical action. It is widely believed that the Weyl symmetry is violated by radiative corrections, thereby giving rise to a nonvanishing value of the cosmological constant at the quantum level. However, we conjecture that the fake Weyl symmetry is kept even in the quantum regime owing to its fakeness. Our conjecture seems to be consistent with the fact that the fake Weyl symmetry has an indentially vanishing Noether current. We wish to consider these problems in near future.

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